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NON-LINEAR THEORY FOR ELASTIC SPATIAL RODS

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Abstract—Starting with a three-dimensional non-linear formulation of elasticity and employing the variational procedure together with a suitable and proper strain and displacement field, general forms of kinematical relations, equilibrium equations, and a set of one-dimensional constitutive equations have been derived for spatial rods undergoing small finite deformations. These equations can be employed to study different types of spatial rod elements for different geometries, loading, and cross-sectional shapes. The formulations take account of the influence of the loading behavior, either conservative or non-conservative loading. The corresponding equations for the classical rod theory are extracted from the derived, general non-linear equations.

1. INTRODUCTION

The non-linear theory of elastic spatial rods permits an approach to the solution of a series of important problems which usually do not arise in classical theory. Problems such as the stability, large deflection, and post-buckling analysis of rods can be studied in this context. In general, there are two sources of non-linearity in structural problems: (1) geometric non-linearity, which occurs when deformations are of such magnitude that their influence in equilibrium considerations cannot be ignored; (2) material non-linearity, which occurs when the stress–strain relations of the structural materials are non-linear. This study is devoted to a procedure for handling certain non-linearities in the rod geometry.

Published works in the field of non-linear analysis of rods are to some extent non-general. In some cases, the formulations presented are limited to a specific kind of rod shape, such as beam, curved beam, or pretwisted rod. In some cases, only the non-linear terms in the strains are taken into account, and the non-linearities in curvatures are ignored. In many cases, the shearing deformations are neglected and the effect of the loading behavior due to the displacement field is not included (see, for example, Pan, 1962a,b; Eisley, 1963; Rosen and Friedmann, 1979; Reissner, 1983, 1984, 1985).

In what follows, the formulations in the non-linear theory of elasticity have been used to set up the kinematical relations for spatial rods. Kirchhoff's assumption has been used, together with a suitable displacement and strain field variation, taking into account the non-linearities in strains and curvatures. The principle of minimum potential energy has been used to derive a system of non-linear equilibrium differential equations and also a form of one-dimensional constitutive relations. The different kinds of loading behavior due to the displacement field have been studied and are incorporated in the formulations. The classical theory of spatial rods has also been derived.

2. GOVERNING EQUATIONS

Consider a volumetric element of a spatial rod as shown in Fig. 1. The orthogonal curvilinear coordinate system $\alpha_1\alpha_2\alpha_3$ is chosen as the natural coordinates of the center line of the rod, with **t** the unit tangential vector, **n** the unit normal vector and **b** the binormal unit vector of the center line in the undeformed state. To derive the governing equations in this curvilinear coordinate system, the position of every point must be expressed in terms of α_1, α_2, and s, where s is the coordinate in the α_3 direction. In this coordinate system the

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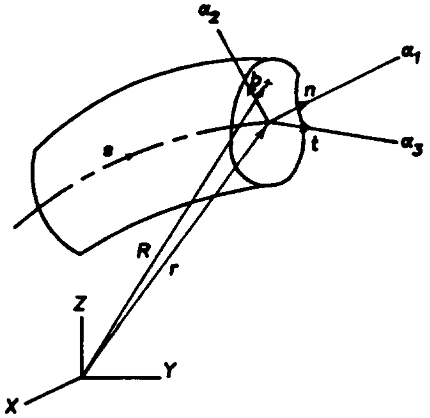

Fig. 1. Global and natural coordinate systems of an undeformed rod.

non-linear strains ($\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31}$) at any point of the spatial rod can be written in terms of the linear strains (e_{ij}) and rotations ω_i, in the form (Novozhilov, 1963):

$$\begin{aligned}
\varepsilon_{11} &= e_{11}+\tfrac{1}{2}(e_{11}^2+(\tfrac{1}{2}e_{12}+\omega_3)^2+(\tfrac{1}{2}e_{13}-\omega_2)^2)\\
\varepsilon_{22} &= e_{22}+\tfrac{1}{2}(e_{22}^2+(\tfrac{1}{2}e_{23}+\omega_1)^2+(\tfrac{1}{2}e_{21}-\omega_3)^2)\\
\varepsilon_{33} &= e_{33}+\tfrac{1}{2}(e_{33}^2+(\tfrac{1}{2}e_{31}+\omega_2)^2+(\tfrac{1}{2}e_{32}-\omega_1)^2)\\
\gamma_{12} &= e_{22}+e_{11}(\tfrac{1}{2}e_{12}-\omega_3)+e_{22}(\tfrac{1}{2}e_{12}+\omega_3)+(\tfrac{1}{2}e_{13}-\omega_2)(\tfrac{1}{2}e_{23}+\omega_1)\\
\gamma_{23} &= e_{23}+e_{22}(\tfrac{1}{2}e_{23}-\omega_1)+e_{33}(\tfrac{1}{2}e_{23}+\omega_1)+(\tfrac{1}{2}e_{21}-\omega_3)(\tfrac{1}{2}e_{31}+\omega_2)\\
\gamma_{31} &= e_{31}+e_{33}(\tfrac{1}{2}e_{31}-\omega_2)+e_{11}(\tfrac{1}{2}e_{31}+\omega_2)+(\tfrac{1}{2}e_{32}-\omega_1)(\tfrac{1}{2}e_{12}+\omega_3).
\end{aligned} \tag{1}$$

If u, v and w designate the displacement components of this volumetric element in the coordinate system $\alpha_1\alpha_2\alpha_3$, it is proved in the linear theory of elasticity by Novozhilov (1963) that the linear strains e_{ij}, in this orthogonal curvilinear coordinate system can be written as

$$\begin{aligned}
e_{11} &= \frac{1}{H_1}\frac{\partial u}{\partial \alpha_1}+\frac{1}{H_1H_2}\frac{\partial H_1}{\partial \alpha_2}v+\frac{1}{H_1H_3}\frac{\partial H_1}{\partial \alpha_3}w\\
e_{22} &= \frac{1}{H_2}\frac{\partial v}{\partial \alpha_2}+\frac{1}{H_2H_3}\frac{\partial H_2}{\partial \alpha_3}w+\frac{1}{H_2H_1}\frac{\partial H_2}{\partial \alpha_1}u\\
e_{33} &= \frac{1}{H_3}\frac{\partial w}{\partial \alpha_3}+\frac{1}{H_3H_1}\frac{\partial H_3}{\partial \alpha_1}u+\frac{1}{H_3H_2}\frac{\partial H_3}{\partial \alpha_2}v\\
e_{12} &= \frac{H_2}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v}{H_2}\right)+\frac{H_1}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{H_1}\right)\\
e_{23} &= \frac{H_3}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{w}{H_3}\right)+\frac{H_2}{H_3}\frac{\partial}{\partial \alpha_3}\left(\frac{v}{H_2}\right)\\
e_{31} &= \frac{H_1}{H_3}\frac{\partial}{\partial \alpha_3}\left(\frac{u}{H_1}\right)+\frac{H_3}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{w}{H_3}\right)
\end{aligned} \tag{2}$$

where H_1, H_2 and H_3 are certain coefficients defined as

$$H_1 = \sqrt{\left(\frac{\partial X}{\partial \alpha_1}\right)^2 + \left(\frac{\partial Y}{\partial \alpha_1}\right)^2 + \left(\frac{\partial Z}{\partial \alpha_1}\right)^2}$$

$$H_2 = \sqrt{\left(\frac{\partial X}{\partial \alpha_2}\right)^2 + \left(\frac{\partial Y}{\partial \alpha_2}\right)^2 + \left(\frac{\partial Z}{\partial \alpha_2}\right)^2}$$

$$H_3 = \sqrt{\left(\frac{\partial X}{\partial \alpha_3}\right)^2 + \left(\frac{\partial Y}{\partial \alpha_3}\right)^2 + \left(\frac{\partial Z}{\partial \alpha_3}\right)^2}. \quad (3)$$

Similarly, the angles of rotation in the same orthogonal curvilinear coordinates can be written in terms of the displacements and the coefficients H_1, H_2 and H_3 as:

$$2\omega_1 = \frac{1}{H_2 H_3}\left[\frac{\partial}{\partial \alpha_2}(H_3 w) - \frac{\partial}{\partial \alpha_3}(H_2 v)\right]$$

$$2\omega_2 = \frac{1}{H_3 H_1}\left[\frac{\partial}{\partial \alpha_3}(H_1 u) - \frac{\partial}{\partial \alpha_1}(H_3 w)\right]$$

$$2\omega_3 = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \alpha_1}(H_2 v) - \frac{\partial}{\partial \alpha_2}(H_1 u)\right]. \quad (4)$$

If the position vector of any volumetric element of rod, $\mathbf{R}$, is expressed in terms of the global coordinates XYZ and the local coordinate system $\alpha_1\alpha_2\alpha_3$ as

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = \mathbf{r} + \alpha_1 \mathbf{n} + \alpha_2 \mathbf{b} \quad (5)$$

where $\mathbf{r}$ is the position vector of the point where the center line meets the perpendicular cross-sectional plane (Fig. 1), then by considering the relations (3) and (5) the coefficients H_1, H_2 and H_3 are defined as

$$H_1 = \frac{\partial \mathbf{R}}{\partial \alpha_1}, \quad H_2 = \frac{\partial \mathbf{R}}{\partial \alpha_2}, \quad H_3 = \frac{\partial \mathbf{R}}{\partial \alpha_3}. \quad (6)$$

Based on Frenet relations for a spatial curve (Thomas, 1965)

$$\frac{d\mathbf{t}}{ds} = K\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - K\mathbf{t}, \quad \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} \quad (7)$$

where K and τ are the curvature and the twist of the center line of the rod element in the undeformed state, respectively. Utilizing (6) and (7) it can be proved that:

$$H_1 = 1.0, \quad H_2 = 1.0, \quad H_3 = \sqrt{(1 - K\alpha_1^2)^2 + (\alpha_2^2 + \alpha_1^2)\tau^2} \quad (8)$$

and also

$$\frac{\partial H_1}{\partial \alpha_2} = \frac{\partial H_1}{\partial \alpha_3} = \frac{\partial H_2}{\partial \alpha_1} = \frac{\partial H_2}{\partial \alpha_3} = 0$$

$$\frac{\partial H_3}{\partial \alpha_1} = \frac{1}{H_3}(-K + K^2\alpha_1 + \alpha_1\tau^2), \quad \frac{\partial H_3}{\partial \alpha_2} = \frac{1}{H_3}(\alpha_2\tau^2). \quad (9)$$

Since the dimensions related to the α_1 and α_2 coordinates are usually much smaller than $1/K$ and $1/\tau$, it can be concluded that for thin rods, the following approximations can be made:

$$H_3 \approx 1.0$$
$$\frac{\partial H_3}{\partial \alpha_1} \approx -K, \quad \frac{\partial H_3}{\partial \alpha_2} \approx 0. \tag{10}$$

We can now write the linear strain components and the rotation angles in terms of the displacements and the curvature by using the relations in (2), (4), (8), (9) and (10) in the following form

$$e_{11} = \frac{\partial u}{\partial \alpha_1}, \quad e_{22} = \frac{\partial v}{\partial \alpha_2}, \quad e_{33} = \frac{\partial w}{\partial s} - Ku$$
$$e_{12} = \frac{\partial u}{\partial \alpha_2} + \frac{\partial v}{\partial \alpha_1}, \quad e_{23} = \frac{\partial v}{\partial s} + \frac{\partial w}{\partial \alpha_2}, \quad e_{13} = \frac{\partial w}{\partial \alpha_1} + \frac{\partial u}{\partial s} + Kw$$
$$2\omega_1 = \frac{\partial w}{\partial \alpha_2} - \frac{\partial v}{\partial s}, \quad 2\omega_2 = \frac{\partial u}{\partial s} - \frac{\partial w}{\partial \alpha_1} + Kw, \quad 2\omega_3 = \frac{\partial v}{\partial \alpha_1} - \frac{\partial u}{\partial \alpha_2}. \tag{11}$$

Now, the non-linear strain components in (1) can be written in terms of displacement components and the curvature of the point of the rod as follows:

$$\varepsilon_{33} = \frac{\partial w}{\partial s} - Ku + \frac{1}{2}\left[\left(\frac{\partial w}{\partial s} - Ku\right)^2 + \left(\frac{\partial u}{\partial s} + Kw\right)^2 + \left(\frac{\partial v}{\partial s}\right)^2\right]$$
$$\gamma_{13} = \frac{\partial u}{\partial s} + Kw + \frac{\partial w}{\partial \alpha_1} + \frac{\partial u}{\partial \alpha_1}\left(\frac{\partial u}{\partial s} + Kw\right) + \frac{\partial v}{\partial \alpha_1}\frac{\partial v}{\partial s}$$
$$\gamma_{23} = \frac{\partial w}{\partial \alpha_2} + \frac{\partial v}{\partial s} + \frac{\partial v}{\partial \alpha_2}\frac{\partial v}{\partial s} + \frac{\partial u}{\partial \alpha_2}\left(\frac{\partial u}{\partial s} + Kw\right). \tag{12}$$

Note that for a slender rod the other strain components ε_{11}, ε_{22} and γ_{12} are small and usually negligible as compared to the quantities ε_{33}, γ_{13} and γ_{23}. Also, the first term in the bracket in the right hand side of the expression for ε_{33} can be neglected as compared to its first power outside the bracket. This assumption corresponds to a related assumption made in the non-linear theory of straight bars (see Dym and Shames, 1973). Utilizing Kirchhoff's assumptions (see for example, Novozhilov, 1963) we may write the displacements at every point within the cross-section in terms of the displacement components of the center line u, v and w and the position α_i and rotation angle of the cross-section Θ_i in the form of

$$u = \bar{u} - \alpha_2\Theta_3$$
$$v = \bar{v} + \alpha_1\Theta_3$$
$$w = \bar{w} - \alpha_1\Theta_2 + \alpha_2\Theta_1. \tag{13}$$

$\bar{u}$, $\bar{v}$, $\bar{w}$, Θ_1, Θ_2 and Θ_3 are functions of s only. The mathematical meanings of Θ_1 and Θ_2 are $\partial w/\partial\alpha_2$ and $-\partial w/\partial\alpha_1$ at $\alpha_1 = \alpha_2 = 0$, respectively. Also, Θ_3 is $-\partial u/\partial\alpha_2$ or $\partial v/\partial\alpha_1$ at $\alpha_1 = \alpha_2 = 0$. Inserting the above approximations for displacements in eqns (12) and ignoring the comparatively small differential terms of higher orders, the three dominant strain components at every point of the rod can be written as

$$\varepsilon_{33} = \varepsilon_3 - \alpha_1 k_2 + \alpha_2 k_1$$
$$\gamma_{13} = \gamma_1 - \alpha_2 k_{31}$$
$$\gamma_{23} = \gamma_2 + \alpha_1 k_{32} \tag{14}$$

where ε_3 is the axial strain, γ_1 and γ_2 are the shear strains, k_1 and k_2 are the curvature changes, and k_{31} and k_{32} are the changes of pretwist of the center line of the rod, and they are in the form of

$$\varepsilon_3 = \underline{\frac{\partial \bar{w}}{\partial s} - K\bar{u}} + \frac{1}{2}\left(\left(\frac{\partial \bar{u}}{\partial s}\right)^2 + \left(\frac{\partial \bar{v}}{\partial s}\right)^2 + K^2\bar{w}^2\right) + K\bar{w}\frac{\partial \bar{u}}{\partial s}$$

$$\gamma_1 = \underline{\frac{\partial \bar{u}}{\partial s} + K\bar{w} - \Theta_2} + \Theta_3\frac{\partial \bar{v}}{\partial s}$$

$$\gamma_2 = \underline{\frac{\partial \bar{v}}{\partial s} + \Theta_1} - \Theta_3\frac{\partial \bar{u}}{\partial s} - K\Theta_3\bar{w}$$

$$k_1 = \underline{\frac{\partial \Theta_1}{\partial s} + K\Theta_3} - \frac{\partial \bar{u}}{\partial s}\frac{\partial \Theta_3}{\partial s} + K^2\bar{w}\Theta_1 + K\Theta_1\frac{\partial \bar{u}}{\partial s} - K\bar{w}\frac{\partial \Theta_3}{\partial s}$$

$$k_2 = \underline{\frac{\partial \Theta_2}{\partial s}} + K^2\Theta_2\bar{w} + K\Theta_2\frac{\partial \bar{u}}{\partial s} - \frac{\partial \bar{v}}{\partial s}\frac{\partial \Theta_3}{\partial s}$$

$$k_{31} = \underline{\frac{\partial \Theta_3}{\partial s} - K\Theta_1}, \quad k_{32} = \underline{\frac{\partial \Theta_3}{\partial s}}. \tag{15}$$

The relations given in (14) and (15) are fairly general non-linear kinematical relations for spatial curved and pretwisted rods undergoing small deformations. The underlined parts of the equations are those applied to classical (linear) rod theory. In order to derive the equilibrium and the one-dimensional constitutive relations as well as the form of the boundary conditions one can use the total potential energy of an element of the rod as the functional of the problem. The variational procedures can then be used as a mathematical tool to find the stationary values of this functional. Here, the total potential energy of an element of the rod is composed of the strain energy and the work done by external forces; it can be written as

$$\text{Total Potential Energy} = \pi = \pi_s + \pi_e \tag{16}$$

where

$$\text{Strain Energy} = \pi_s = \iiint (E\varepsilon_{33}^2 + G\gamma_{13}^2 + G\gamma_{23}^2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}s \tag{17}$$

$$\begin{aligned}\text{External Work} = \pi_e = &-\int (F_1\bar{u} + F_2\bar{v} + F_3\bar{w} + G_1\Theta_1 + G_2\Theta_2 + G_3\Theta_3)\,\mathrm{d}s \\ &- \sum_{k=i,j} [\bar{Q}_1\bar{u} + \bar{Q}_2\bar{v} + \bar{Q}_3\bar{w} + \bar{M}_1\Theta_1 + \bar{M}_2\Theta_2 + \bar{M}_3\Theta_3]_{s=s_k}\end{aligned} \tag{18}$$

in which F_1, F_2 and F_3 are the external distributed forces, and G_1, G_2 and G_3 are the distributed moments along the length of an element of a rod from s_i to s_j. $\bar{Q}_1$, $\bar{Q}_2$ and $\bar{Q}_3$ are the internal forces, and $\bar{M}_1$, $\bar{M}_2$ and $\bar{M}_3$ are the internal moments at the ends of the rod

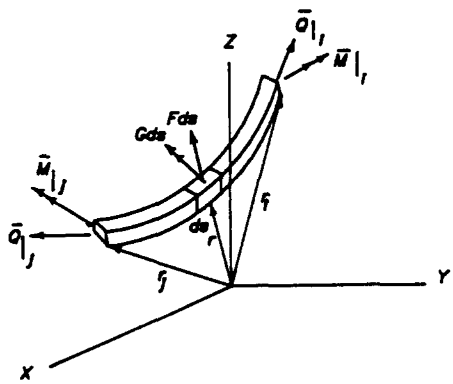

Fig. 2. Internal and external forces on an element of a spatial rod.

(Fig. 2). For the rod to be in equilibrium, the variation of the total potential energy of the element must be zero, that is:

$$\delta\pi = \delta(\pi_s + \pi_e) = \delta\pi_s + \delta\pi_e = 0. \tag{19}$$

The variation in the strain energy can be written as:

$$\delta\pi_s = \iiint (\sigma_{33}\delta\varepsilon_{33} + \sigma_{13}\delta\gamma_{13} + \sigma_{23}\delta\gamma_{23})\, d\alpha_1\, d\alpha_2\, ds \tag{20}$$

in which

$$\begin{aligned} \delta\varepsilon_{33} &= \delta\varepsilon_3 - \alpha_1\delta k_2 + \alpha_2\delta k_1 \\ \delta\gamma_{13} &= \delta\gamma_1 - \alpha_2\delta k_{31} \\ \delta\gamma_{23} &= \delta\gamma_2 + \alpha_1\delta k_{32} \\ \sigma_{33} &= E\varepsilon_{33}, \quad \sigma_{13} = G\gamma_{13}, \quad \sigma_{23} = G\gamma_{23} \end{aligned} \tag{21}$$

where σ_{ij} are the stress components. Carrying out the integration over the cross-section of the rod, i.e. integrating over α_1 and α_2, we obtain

$$\delta\pi_s = \int (Q_1\delta\gamma_1 + Q_2\delta\gamma_2 + Q_3\delta\varepsilon_3 + M_1\delta k_1 + M_2\delta k_2 + M_{31}\delta k_{31} + M_{32}\delta k_{32})\, ds \tag{22}$$

in which

$$\begin{aligned} (Q_3, M_1, M_2) &= \iint (1, \alpha_2, -\alpha_1)\sigma_{33}\, d\alpha_1\, d\alpha_2 \\ (Q_1, Q_2, M_{31}, M_{32}) &= \iint (\sigma_{13}, \sigma_{23}, -\alpha_2\sigma_{13}, \alpha_1\sigma_{23})\, d\alpha_1\, d\alpha_2 \end{aligned} \tag{23}$$

where (Q_1, Q_2, Q_3) and (M_1, M_2) are the sectional internal forces and moments, respectively. M_{31} and M_{32} are the cross-sectional torques.

Using the relations (15), (18), (19) and (22), we obtain the following equilibrium equations, the so-called Euler equations related to the functional:

$$\underline{\frac{\mathrm{d}}{\mathrm{d}s}\Big[Q_1}-\Theta_3 Q_2+\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}+K\bar{w}\right)Q_3+K\Theta_2 M_2-\left(\frac{\mathrm{d}\Theta_3}{\mathrm{d}s}-K\Theta_1\right)M_1\Big]\underline{+KQ_3+F_1=0}$$

$$\underline{\frac{\mathrm{d}}{\mathrm{d}s}\Big(Q_2}+\Theta_3 Q_1+\frac{\mathrm{d}\bar{v}}{\mathrm{d}s}Q_3-\frac{\mathrm{d}\Theta_3}{\mathrm{d}s}M_2\Big)\underline{+F_2=0}$$

$$\underline{\frac{\mathrm{d}Q_3}{\mathrm{d}s}}-KQ_1+KQ_2\Theta_3-\left(K^2\bar{w}+K\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}\right)Q_3-K^2\Theta_2 M_2+\left(K\frac{\mathrm{d}\Theta_3}{\mathrm{d}s}-K^2\Theta_1\right)M_1\underline{+F_3=0}$$

$$\underline{\frac{\mathrm{d}M_1}{\mathrm{d}s}+KM_{31}-Q_2+G_1}-\left(K^2\bar{w}+K\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}\right)M_1\underline{=0}$$

$$\underline{\frac{\mathrm{d}M_2}{\mathrm{d}s}+Q_1+G_2}-\left(K^2\bar{w}+K\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}\right)M_2\underline{=0}$$

$$\underline{\frac{\mathrm{d}}{\mathrm{d}s}\Big[(M_{31}+M_{32})}-\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}+K\bar{w}\right)M_1-\frac{\mathrm{d}\bar{v}}{\mathrm{d}s}M_2\Big]+\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}+K\bar{w}\right)Q_2-\frac{\mathrm{d}\bar{v}}{\mathrm{d}s}Q_1\underline{-KM_1+G_3=0.}$$

(24)

Again, the underlined parts of the equations are those applied to the classical rod theory.

The natural boundary conditions for the above differential equations at end i may be derived as

$$\left[Q_1-\Theta_3 Q_2+\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}+k\bar{w}\right)Q_3+K\Theta_2 M_2-\left(\frac{\mathrm{d}\Theta_3}{\mathrm{d}s}-K\Theta_1\right)M_1\right]_{s=s_i}=\bar{Q}_1|_i$$

$$\left(Q_2+\Theta_3 Q_1+\frac{\mathrm{d}\bar{v}}{\mathrm{d}s}Q_3-\frac{\mathrm{d}\Theta_3}{\mathrm{d}s}M_2\right)_{s=s_i}=\bar{Q}_2|_i$$

$$\left[(M_{31}+M_{32})-\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}+K\bar{w}\right)M_1-\frac{\mathrm{d}\bar{v}}{\mathrm{d}s}M_2\right]_{s=s_i}=\bar{M}_3|_i$$

$$(Q_3)_{s=s_i}=\bar{Q}_3|_i,\quad (M_1)_{s=s_i}=\bar{M}_1|_i,\quad (M_2)_{s=s_i}=\bar{M}_2|_i \tag{25}$$

and the geometric boundary condition is

$$(\bar{u},\bar{v},\bar{w},\Theta_1,\Theta_2,\Theta_3)_{s=s_i}=(\bar{u},\bar{v},\bar{w},\Theta_1,\Theta_2,\Theta_3)|_i. \tag{26}$$

To derive the one-dimensional constitutive equations (the moment–curvature and force–strain relationships), one can use the relations given in (14), (21) and (23). The matrix forms of the one-dimensional heuristic constitutive equations are

$$\begin{Bmatrix} Q_3 \\ M_1 \\ M_2 \end{Bmatrix}=\begin{pmatrix} A_E & S_{1E} & -S_{2E} \\ S_{1E} & I_{1E} & -I_{12E} \\ -S_{2E} & -I_{12E} & I_{2E} \end{pmatrix}\begin{Bmatrix} \varepsilon_3 \\ k_1 \\ k_2 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ M_{31} \\ M_{32} \end{Bmatrix}=\begin{pmatrix} A_G & 0 & S_{1G} & 0 \\ 0 & A_G & 0 & S_{2G} \\ -S_{1G} & 0 & I_{1G} & 0 \\ 0 & S_{2G} & 0 & I_{2G} \end{pmatrix}\begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ k_{31} \\ k_{32} \end{Bmatrix} \tag{27}$$

in which

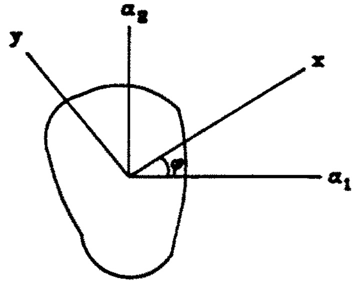

Fig. 3. Natural and principal local axes of the rod cross-section.

$$(A_E, S_{1E}, S_{2E}, I_{1E}, I_{2E}, I_{12E}) = \iint (1, \alpha_2, \alpha_1, \alpha_2^2, \alpha_1^2, \alpha_1\alpha_2) E \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2$$

$$(A_G, S_{1G}, S_{2G}, I_{1G}, I_{2G}) = \iint (1, \alpha_2, \alpha_1, \alpha_2^2, \alpha_1^2) G \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2. \tag{28}$$

In the above equations the elasticity constants E and G can be variable over the cross-section with respect to α_1 and α_2.

Equations (15), (24), (25) and (27) are the basic governing equations for small deformation theory of elastic spatial rods in natural coordinate systems. If one sets the curvature of the rod equal to zero; that is, $K_1 = K_2 = K_3 = 0$, the governing equations for straight beams are derived. In the same way, by inserting $K_1 = K_2 = 0$ in the above-mentioned equations the governing equations for pretwisted rods are derived. The resulting equations are the same as those derived by Reissner (1983, 1985). For another test, one can obtain the formulations for the classical theory of spatial rods.

3. CLASSICAL THEORY OF SPATIAL RODS

Let xy be the principal axes of the cross-section of the rod making an angle Φ with the natural axes $\alpha_1\alpha_2$, as shown in Fig. 3. Using the rotation matrix for the $xy\alpha_3$ coordinate system in $\alpha_1\alpha_2\alpha_3$, and employing the underlined terms in eqn (15), the kinematical relations in the local coordinate system can be written in the form

$$\{\varepsilon^{\mathrm{L}}\} = \frac{\mathrm{d}}{\mathrm{d}s}\{\bar{u}^{\mathrm{L}}\} + [K]\{\bar{u}^{\mathrm{L}}\} + [J]\{\Theta^{\mathrm{L}}\}$$

$$\{k^{\mathrm{L}}\} = \frac{\mathrm{d}}{\mathrm{d}s}\{\hat{\Theta}^{\mathrm{L}}\} + [K_1]\{\Theta^{\mathrm{L}}\} \tag{29}$$

where

$$\{\varepsilon^{\mathrm{L}}\} = \begin{Bmatrix} \gamma_x \\ \gamma_y \\ \gamma_3 \end{Bmatrix}, \quad \{\bar{u}^{\mathrm{L}}\} = \begin{Bmatrix} \bar{u}^{\mathrm{L}} \\ \bar{v}^{\mathrm{L}} \\ \bar{w}^{\mathrm{L}} \end{Bmatrix}, \quad \{k^{\mathrm{L}}\} = \begin{Bmatrix} k_x \\ k_y \\ k_{3x} \\ k_{3y} \end{Bmatrix}, \quad \{\hat{\Theta}^{\mathrm{L}}\} = \begin{Bmatrix} \Theta_x \\ \Theta_y \\ \Theta_3 \\ \Theta_3 \end{Bmatrix}, \quad \{\Theta^{\mathrm{L}}\} = \begin{Bmatrix} \Theta_x \\ \Theta_y \\ \Theta_3 \end{Bmatrix}.$$

The superscript L means that the value in the local $(xy\alpha_3)$ coordinate system has been used. The matrix $[K]$ is the curvature matrix in the local coordinate system, that is

$$[K] = \begin{pmatrix} 0 & -K_z & K_y \\ K_z & 0 & -K_x \\ -K_y & K_x & 0 \end{pmatrix} \tag{30}$$

where $K_x = K \sin \Phi$, $K_y = K \cos \Phi$ and $K_z = \mathrm{d}\Phi/\mathrm{d}s$. The matrices $[J]$ and $[K_1]$ in eqn (29) are given as

$$[J] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [K_1] = \begin{pmatrix} 0 & -K_z & K_y \\ K_z & 0 & -K_x \\ -K_y \cos^2 \Phi & K_x \cos^2 \Phi & 0 \\ -K_y \sin^2 \Phi & K_x \sin^2 \Phi & 0 \end{pmatrix}. \tag{31}$$

Using the underlined terms in eqn (24) the equilibrium equations for classical rod theory in the local $xy\alpha_3$ coordinate system can be simplified to the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\{Q^L\} + [K]\{Q^L\} + \{F^L\} = \{0\}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\{M^L\} - [K_1]^T\{\hat{M}^L\} + [J]\{Q^L\} + \{G^L\} = \{0\} \tag{32}$$

wherein

$$\{Q^L\} = \begin{Bmatrix} Q_x \\ Q_y \\ Q_3 \end{Bmatrix}, \quad \{F^L\} = \begin{Bmatrix} F_x \\ F_y \\ F_3 \end{Bmatrix}, \quad \{G^L\} = \begin{Bmatrix} G_x \\ G_y \\ G_3 \end{Bmatrix}, \quad \{M^L\} = \begin{Bmatrix} M_x \\ M_y \\ M_{3x} + M_{3y} \end{Bmatrix},$$

$$\{\hat{M}^L\} = \begin{Bmatrix} M_x \\ M_y \\ M_{3x} \\ M_{3y} \end{Bmatrix}.$$

By utilizing the constitutive relations given in (27), the moment-curvature and force-strain relations for the rod in classical theory can be written in the local coordinate system (assuming E and G are constants over the cross-section) as

$$\{Q^L\} = [A]\{\varepsilon^L\}$$

$$\{\hat{M}^L\} = [I]\{k^L\} \tag{33}$$

where

$$[A] = \begin{pmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{pmatrix}, \quad [I] = \begin{pmatrix} EI_x & 0 & 0 & 0 \\ 0 & EI_y & 0 & 0 \\ 0 & 0 & GI_x & 0 \\ 0 & 0 & 0 & GI_y \end{pmatrix}.$$

The general formulations developed can be employed for different kinds of spatial rod elements such as straight beams, curved beams and pretwisted rods. For beams, we insert $K_x = K_y = K_z = 0$ in the formulations. In the same way, for curved beams we insert $K_z = 0$, and for pretwisted rods we let $K_x = K_y = 0$. The special equations derived in this way are in full agreement with those obtained for classical rod theory (Love, 1944; also see, for example, Tabarrok *et al.*, 1988; Farshad *et al.*, 1989; Banan *et al.*, 1989).

4. BEHAVIOR OF APPLIED LOADING DUE TO DEFORMATION FIELD

The deformations in non-linear elasticity theory cannot be assumed infinitesimal; therefore, the magnitudes and the directions of the applied external forces in the deformed state of equilibrium are not the same as those in the undeformed state. The transformation matrix from the undeformed state to the deformed state based on non-linear theory of elasticity (Novozhilov, 1963) can be written as

$$[T] = \begin{pmatrix} 1+e_{11} & e_{12}+\omega_3 & e_{13}-\omega_2 \\ e_{12}-\omega_3 & 1+e_{22} & e_{23}+\omega_1 \\ e_{13}+\omega_2 & e_{23}-\omega_1 & 1+e_{33} \end{pmatrix}. \tag{34}$$

The above matrix can be simplified for the spatial rods if one can insert instead of e_{ij} and ω_j the terms of displacements and rotation angles from relations (11) and (13). It becomes

$$[T] = \begin{bmatrix} 1.0 & \Theta_3 & -\Theta_2 \\ -\Theta_3 & 1.0 & \Theta_1 \\ \dfrac{d\bar{u}}{ds} + K\bar{w} & \dfrac{d\bar{v}}{ds} & 1.0 \end{bmatrix}. \tag{35}$$

The equilibrium equations (25) have been written in the deformed state but as related to the undeformed directions. Therefore, the external applied loads $\{F\}$ and $\{G\}$ must be specified correspondingly. To make distinctions between the loadings at the undeformed and the deformed state let $\{\hat{F}\}$ and $\{\hat{G}\}$ represent the loads in the undeformed state in undeformed directions and $\{\bar{F}\}$ and $\{\bar{G}\}$ represent the loads in the deformed state in the deformed directions. It must be mentioned that $\{F\}$ and $\{G\}$ are usually functions of the displacement field. Three different kinds of load behavior due to the deformation field are discussed here. These are the constant direction load, the follower load, and the polar load.

4.1. *Constant direction load*

In this case, $\{\hat{F}\}$ and $\{\hat{G}\}$ are given and have constant directions; therefore,

$$\{\bar{F}\} = [T]\{\hat{F}\}, \quad \{\bar{G}\} = [T]\{\hat{G}\}. \tag{36}$$

Utilizing the following relations

$$\{F\} = [T]^T\{\bar{F}\} \quad \text{and} \quad \{G\} = [T]^T\{\bar{G}\}$$

we have

$$\{F\} = \{\hat{F}\} \quad \text{and} \quad \{G\} = \{\hat{G}\}. \tag{37}$$

This means that as far as the directions are concerned the external loads must be directly inserted in the equilibrium equations, but for their magnitudes, they are functions of the displacement field.

4.2. *Follower load*

Because of the nature of this kind of loading

$$\{\hat{F}\} = \{\bar{F}\}, \quad \{\hat{G}\} = \{\bar{G}\}. \tag{38}$$

Utilizing the following relations

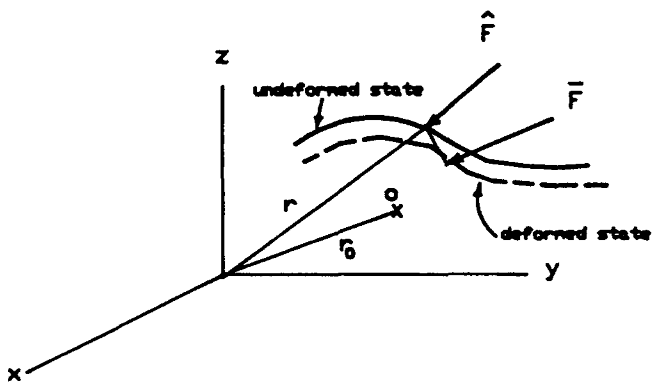

Fig. 4. Polar loading behavior due to displacement field.

$$\{F\} = [T]^{\mathrm{T}}\{\bar{F}\},\quad \{G\} = [T]^{\mathrm{T}}\{\bar{G}\}$$

we have

$$\{F\} = [T]^{\mathrm{T}}\{F\},\quad \{G\} = [T]^{\mathrm{T}}\{\hat{G}\}. \tag{39}$$

Therefore, for this kind of loading the directions of the applied external loads must be transformed into the undeformed directions using the transformation matrix knowing that their magnitudes are functions of the displacement field.

4.3. *Polar load*

In this kind of loading the applied external forces are always directed to a point called pole; if they are of moment type, then the moment axis goes through the pole. By considering a spatial road as in Fig. 4, if the vectors $\mathbf{r}_0$, $\mathbf{r}$ and $\mathbf{u}$ are defined in natural coordinates $\alpha_1\alpha_2\alpha_3$, we will have

$$\{F\} = \|\hat{F}\|\,\frac{(\mathbf{r}_0-\mathbf{r}-\mathbf{u})}{\|\mathbf{r}_0-\mathbf{r}-\mathbf{u}\|}$$

$$\{G\} = \|\hat{G}\|\,\frac{(\mathbf{r}_0-\mathbf{r}-\mathbf{u})}{\|\mathbf{r}_0-\mathbf{r}-\mathbf{u}\|} \tag{40}$$

where $\{u\} = \{u, v, w\}^{\mathrm{T}}$, $\mathbf{r}$ is the position vector of the external applied load in the unreformed state and $\mathbf{r}_0$ is the position vector of the pole all expressed in the natural coordinate system. In the above equations the load is assumed to act toward the pole; if it acts away from the pole it must be multiplied by -1 and its magnitude may be a function of the displacement field.

5. CONCLUSION

A small finite deformation theory for spatial rods was presented. Different kinds of loading, conservative or non-conservative, were discussed and formulated. As a special case, the classical rod theory was derived. The formulations are in a systematic form and may be applied to different rod elements such as straight and curved beams and pretwisted rods. One of the salient features of the formulations is to study the instability of the slender bodies. The formulation can also be used as a basis for use with different numerical techniques to study the non-linear behavior of rods under different loading conditions. In this respect, a theoretical and numerical finite element stability analysis of spatial rod systems has been developed which is the subject of another paper by Karami *et al.* (1990).

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